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Extensions of the mild-slope equation

By D. PORTER AND D. J. STAZIKER

Department of Mathematics, The University of Reading, PO Box 220, Whiteknights, Reading, RG6 6AF, UK

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The use of the mild-slope approximation, which is invoked to simplify the problem of linear water wave diffraction–refraction by bed undulations, is reassessed by using a variational method. It is found that smooth approximations to the free surface elevation obtained by using the long-standing mild-slope equation are not consistent with the continuity of mass flow at locations where the bed slope is discontinuous. The use of interfacial jump conditions at such locations significantly improves the accuracy of approximations generated by the mild-slope equation and by the recently derived modified mild-slope equation. The variational principle is also used to produce a generalization of these equations and of the associated jump condition. Numerical results are presented to illustrate the main points of the theory.

1. Introduction

The mild-slope approximation was introduced by Berkhoff (1973) as a way of approximating the refraction and diffraction of linearized surface waves on water of varying quiescent depth. On the basis of the mild-slope assumption, that the relative change in the equilibrium water depth over a wavelength is small, Berkhoff approximated the vertical structure of the fluid motion by using the local flat bed solution corresponding to propagating waves. An averaging process over the fluid depth then removed the vertical coordinate from the governing equations, leaving a boundary value problem posed only in terms of the horizontal coordinates. This simplification may be regarded as an extension of the shallow water approximation. Indeed, the mild-slope equation reduces to the familiar shallow water equation, for time-harmonic motions, if the long wave approximation is invoked. It is, of course, the reduction in the dimension of the problem and therefore in the computational effort required to solve refraction–diffraction problems which gives significance to the mild-slope equation.

Smith & Sprinks (1975) gave another derivation of the mild-slope equation, similar to that of Berkhoff but more succinct, and the mild-slope approximation has since been used to produce other equations which model the effect of bed topography on wave propagation. Kirby (1986), responding to the failure of the mild-slope equation to approximate adequately wave scattering by a ripple bed (consisting of a finite patch of periodic undulations set in an otherwise horizontal bed), gave what is referred to as the extended mild-slope equation. This equation, which was also derived using the vertical averaging technique, applies where the bed profile consists of a rapidly varying small-amplitude component superimposed on a slowly varying component (in the sense of the mild-slope approximation). In particular, Kirby showed that the extended mild-slope equation successfully models the principal resonance found in ripple bed scattering, by comparing numerical calculations with experimental data of Davies & Heathershaw (1984).

More recently, Chamberlain & Porter (1995*a*) have produced the modified mild-slope equation, which contains the original mild-slope equation and its extended version as special cases. We shall need to refer to the derivation of the modified mild-slope equation later and at present need only to remark on its principal features. The equation results from the use of a one-term trial function, based on the propagating wave mode over a flat bed, in a variational principle corresponding to the underlying boundary value problem. An equivalent derivation, using the Galerkin weak form of the boundary value problem, formalizes the vertical averaging procedure used by Berkhoff (1973), Smith & Sprinks (1975) and Kirby (1986). Both the variational and weak form approaches show that the two approximations which together lead to the mild-slope equation, namely the use of the one-term trial function mentioned above and the discarding of terms which are small on the basis of the mild-slope hypothesis, are essentially independent. The second of these approximations is not used in the derivation of the modified mild-slope equation.

Booij (1983) examined the accuracy of the mild-slope equation by carrying out a number of numerical calculations. In particular, he considered the scattering of a plane wave train normally incident on a bedform consisting of a plane sloping section linking two horizontal plane sections. By comparing the amplitude of the reflected wave predicted by the mild-slope equation with that given by solving the unapproximated ('full linear') boundary value problem using the finite element method, Booij concluded that the mild-slope equation can be used in scattering problems with bed slopes up to 1 in 3. We consider Booij's problem in §5.

Despite the marked improvement over the mild-slope equation which it produces for ripple bed scattering, Chamberlain & Porter (1995*a*) found that the modified mild-slope equation performs no better than its predecessor when used in Booij's problem. In the present paper we explain this result and, in the process, show how approximations based on the original mild-slope equation can be significantly improved. Previously, solutions have been derived which are continuous and have continuous first derivatives. These solutions are physically plausible in that they imply continuity of the approximation to the free surface and its slope. However, such solutions are not consistent with the requirement of conservation of mass at locations where the bed profile has a discontinuous slope. We show, by using a variational principle, that solutions of the mild-slope equation and its modified version must satisfy a jump condition where the bed slope is not continuous, in order to ensure continuity of mass flow. This condition, which overrides that of continuity of free surface slope, not only establishes the superiority of the modified mild-slope equation when used to solve Booij's problem, but also improves the accuracy of solutions given by the mild-slope equation.

To show how the overall approximation technique can be developed, and to check the accuracy of those approximations based on the mild-slope hypothesis, we use an extended trial function in the variational principle, incorporating a finite number of terms including the local propagating mode. The corresponding jump condition which ensures mass conservation where the bed slope is discontinuous is also derived. It is found that a similar higher-order approximation derived by Massel (1993) using Galerkin's method is deficient in that mass conservation is again violated.

Results of numerical calculations are given to illustrate the main points of the work.

2. A variational principle

We use Cartesian coordinates (x, y, z) with z measured vertically upwards from the undisturbed free surface and the bed given by $z = -h(x, y)$ where h is a continuous function.

The usual assumptions of linearized water wave theory and the removal of the harmonic time dependence $\exp(-\mathrm{i}\sigma t)$ lead to the familiar equations

$$\left.\begin{aligned} \nabla^2\phi &= 0 \qquad && (-h < z < 0)\,, \\ \phi_z - v\phi &= 0 \qquad && (z = 0)\,, \\ \phi_z + \nabla_h h \cdot \nabla_h \phi &= 0 \qquad && (z = -h)\,, \end{aligned}\right\} \tag{2.1}$$

where $v = \sigma^2/g$, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and $\nabla_h = (\partial/\partial x, \partial/\partial y)$. The free surface elevation is given in terms of the time-independent potential $\phi(x, y, z)$ by

$$\zeta(x, y, t) = \mathrm{Re}\left\{\frac{-\mathrm{i}\sigma}{g}\phi(x, y, 0)\mathrm{e}^{-\mathrm{i}\sigma t}\right\}\,. \tag{2.2}$$

Other conditions satisfied by ϕ, such as those to be applied on lateral boundaries or asymptotically if the fluid extends to infinity, do not immediately concern us.

Chamberlain & Porter (1995*a*) derived the modified mild-slope equation by making use of the functional

$$L(\psi) = \int\!\!\int_D F \mathrm{d}x\, \mathrm{d}y\,,$$

where

$$F = \frac{1}{2}v\left(\psi^2\right)_{z=0} - \frac{1}{2}\int_{-h}^{0} (\nabla\psi)^2\, \mathrm{d}z$$

and D denotes a simply connected domain in the plane $z = 0$, with boundary C. If $\delta\psi$ denotes an arbitrary variation in ψ, we find after some straightforward manipulation that the corresponding first variation in L can be written as

$$\delta L = \int\!\!\int_D \delta F \mathrm{d}x\, \mathrm{d}y + \int_C \boldsymbol{n} \cdot \int_{-h}^{0} \delta\psi \nabla_h \psi \mathrm{d}z\ \mathrm{d}C\,, \tag{2.3}$$

where

$$\delta F = -\left(\delta\psi\left(\psi_z - v\psi\right)\right)_{z=0} + \left(\delta\psi\left(\psi_z + \nabla_h h \cdot \nabla_h \psi\right)\right)_{z=-h} + \int_{-h}^{0} \delta\psi \nabla^2 \psi \mathrm{d}z$$

and $\boldsymbol{n}$ is the outward normal unit vector on C.

It follows that L is stationary for arbitrary variations $\delta\psi$ which vanish on $C \times [-h, 0]$ if and only if $\psi = \phi$ where ϕ satisfies (2.1) in $D \times [-h, 0]$.

Now suppose that we introduce a smooth, simple curve Γ which divides D into two simple connected domains, D_- and D_+, say. Γ may join two points of C or it may lie wholly within D. We readily find that the first variation of the new functional

$$\mathscr{L}(\psi) = \int\!\!\int_{D_-} F \mathrm{d}x\, \mathrm{d}y + \int\!\!\int_{D_+} F \mathrm{d}x\, \mathrm{d}y$$

is given by

$$\delta\mathscr{L} = \int\int_{D_-} \delta F \mathrm{d}x\, \mathrm{d}y + \int\int_{D_+} \delta F \mathrm{d}x\, \mathrm{d}y$$
$$+ \int_C \boldsymbol{n} \cdot \int_{-h}^{0} \delta\psi \nabla_h \psi \mathrm{d}z\, \mathrm{d}C + \int_\Gamma \boldsymbol{n}_\Gamma \cdot \left[\int_{-h}^{0} \delta\psi \nabla_h \psi \mathrm{d}z\right] \mathrm{d}\Gamma\ ,$$

where $\boldsymbol{n}_\Gamma$ denotes the unit normal vector on Γ, directed from D_- to D_+, say. We have also introduced the notation [.] which denotes the jump in the value of the enclosed quantity across the surface $\Gamma \times [-h, 0]$ in the sense that $[\psi] = \psi_- - \psi_+$, using the subscripts $\pm$ in the obvious way.

If we now suppose that $[\psi] = 0$ and consider variations in ψ which vanish on $C \times [-h, 0]$ and satisfy $[\delta\psi] = 0$, we see that $\mathscr{L}$ is stationary at $\psi = \phi$ if and only if ϕ satisfies (2.1) in $D_\pm \times [-h, 0]$ together with

$$[\phi] = 0, \qquad [\boldsymbol{n}_\Gamma \cdot \nabla_h \phi] = 0\ . \tag{2.4}$$

The first element of (2.4) follows by hypothesis and represents continuity of pressure across $\Gamma \times [-h, 0]$. The second element is a consequence of $\delta\mathscr{L} = 0$ and represents continuity of normal velocity, and therefore continuity of mass flow, across $\Gamma \times [-h, 0]$.

Approximations to the solution of (2.1) follow by using the Ritz method, in which ψ is restricted to a particular class of functions and an approximation to the stationary point of $L(\psi)$ or of $\mathscr{L}(\psi)$ is found from within this class. The modified mild-slope equation was derived by Chamberlain & Porter (1995*a*) in this way, by using $L(\psi)$. The key feature of the new functional $\mathscr{L}(\psi)$ is that, in addition to approximating the solution of (2.1), it produces a consistent approximation to the conservation of mass across an interface between two contiguous domains.

More sophisticated variational principles can be devised which have a radiation condition on $C \times [-h, 0]$ as a further natural condition, for example. However, it is sufficient to determine consistent approximations to the interfacial conditions (2.4) and implement these in other conditions which arise.

3. The modified mild-slope equation

Following Chamberlain & Porter (1995*a*), we first approximate $\phi(x, y, z)$ by

$$\psi(x, y, z) = \frac{\mathrm{i}g}{\sigma} w_0(h, z)\phi_0(x, y)\ , \qquad w_0(h, z) = \mathrm{sech}(kh)\cosh(k(z + h))\ , \tag{3.1}$$

where $k = k(x, y)$, the local wavenumber, is the positive, real root of the local dispersion relation

$$v = k \tanh(kh) \tag{3.2}$$

corresponding to the depth $h(x, y)$. The corresponding approximation to the free surface elevation is given by $\zeta(x, y, t) \approx \mathrm{Re}\{\phi_0(x, y)\mathrm{e}^{-\mathrm{i}\sigma t}\}$. Over a flat bed, this ψ is an exact solution of (2.1) with $\nabla_h^2\phi_0 + k^2\phi_0 = 0$.

Using (3.1) in $L(\psi)$, we obtain a new functional $L_1(\phi_0)$, say, and a straightforward calculation gives

$$\delta L_1 = \left(\frac{ig}{\sigma}\right)^2 \left\{\int\int_D G\delta\phi_0 \mathrm{d}x\, \mathrm{d}y + \int_C \delta\phi_0 \boldsymbol{n} \cdot \int_{-h}^{0} w_0 \nabla_h(w_0\phi_0)\mathrm{d}z\, \mathrm{d}C\right\}\ ,$$

where

$$G = \nabla_h \cdot u_0 \nabla_h \phi_0 + \left\{ k^2 u_0 + \int_{-h}^{0} w_0 \nabla_h{}^2 w_0 \mathrm{d}z + \nabla_h h \cdot (w_0 \nabla_h w_0)_{z=-h} \right\} \phi_0 \, .$$

For variations $\delta\phi_0$ which vanish on C, the stationary principle $\delta L_1 = 0$ gives $G = 0$ in D, which is the modified mild-slope equation, expressible in the neater form

$$\nabla_h \cdot u_0 \nabla_h \phi_0 + \left\{ k^2 u_0 + u_1 \nabla_h{}^2 h + u_2 (\nabla_h h)^2 \right\} \phi_0 = 0 \, . \tag{3.3}$$

The coefficients u_0, u_1 and u_2 can be expressed as functions of h, using (3.2) to define $k = k(h)$ implicitly, and are given by

$$u_0(h) = \int_{-h}^{0} w_0^2 \mathrm{d}z, \;\; u_1(h) = \int_{-h}^{0} w_0 \frac{\partial w_0}{\partial h} \mathrm{d}z, \;\; u_2(h) = u_1'(h) - \int_{-h}^{0} \left(\frac{\partial w_0}{\partial h} \right)^2 \mathrm{d}z. \tag{3.4}$$

Some details of the calculation leading to (3.3) may be found in Chamberlain & Porter (1995*a*); explicit expressions for $u_i(h)$, $i = 0, 1, 2$ are given in §4.

Suppose now that (3.1) is used in the functional $\mathscr{L}(\psi)$, with $\phi_0(x, y)$ assumed to be continuous across Γ. If variations $\delta\phi_0$ are considered which vanish on C and satisfy $[\delta\phi_0] = 0$, the natural conditions of the corresponding stationary principle consist of (3.3), holding in $D_- \cup D_+$, together with the jump condition

$$\boldsymbol{n}_\Gamma \cdot \left[\int_{-h}^{0} w_0 \nabla_h (w_0 \phi_0) \, \mathrm{d}z \right] = 0 \, , \tag{3.5}$$

applying at each point of Γ. Since $\nabla_h(w_0\phi_0) = w_0 \nabla_h \phi_0 + \phi_0(\partial w_0/\partial h)\nabla_h h$ and ϕ_0 and h are continuous by assumption, we can arrange (3.5) in the form

$$u_1 \phi_0 \left[\boldsymbol{n}_\Gamma \cdot \nabla_h h \right] + u_0 \left[\boldsymbol{n}_\Gamma \cdot \nabla_h \phi_0 \right] = 0 \tag{3.6}$$

on Γ, using the notation of (3.4).

This jump condition is therefore the form taken by mass conservation at an interface between two domains, when the particular approximation $\phi \approx \psi$ given by (3.1) is used. Continuity of surface elevation (and therefore of pressure) is implied by seeking solutions of (3.3) which are everywhere continuous.

We see that where the bed slope $\nabla_h h$ is continuous, continuity of mass flow is guaranteed by seeking solutions of (3.3) which have continuous first derivatives. However, at locations where the bed slope is discontinuous, mass flow is conserved only if an internal boundary is inserted at that location and $[\phi_0] = 0$ together with (3.6) are imposed across that boundary.

We can, of course, derive (3.6) from (3.3). For, suppose that ϕ_0 and h (and hence u_0, u_1 and u_2) are continuous across Γ but that $\nabla_h h$ is not. Integrating (3.3) over a narrow strip containing Γ and shrinking this strip onto Γ from both sides readily leads to (3.6). This consistency between (3.3) and (3.6) is inevitable, as the calculation we have just described is merely an alternative presentation of that inherent in the variational principle $\delta\mathscr{L} = 0$.

This viewpoint does, however, expose a deficiency in previous calculations involving the mild-slope equation, which is given by deleting the terms $O\left(\nabla_h^2 h, \, (\nabla_h h)^2\right)$ in (3.3) on the basis of the mild-slope approximation $|\nabla_h h| \ll kh$ and is therefore

$$\nabla_h \cdot u_0 \nabla_h \phi_0 + k^2 u_0 \phi_0 = 0 \, . \tag{3.7}$$

The corresponding deletion of the first term in (3.6) would result in an imbalance in the mass flow at every location where the bed slope is discontinuous, and is not necessary to achieve consistency with (3.7). To see this, let $\nabla_h h/kh = O(\epsilon)$, where $\epsilon \ll 1$ and suppose that $\nabla_h \phi_0/k\phi_0 = O(1)$. Then the terms in (3.3) which are ignored to give (3.7) are $O(\epsilon^2)$ relative to the terms which are retained. However, the first term in (3.6) is $O(\epsilon)$ relative to the second and its retention must therefore significantly improve the approximation to ϕ given by (3.7).

It should be noted here that, prior to the derivation of (3.6) by Chamberlain & Porter (1995*a*), the mild-slope equation (3.7) had been obtained directly, rather than as a special case of (3.3), the terms $O\left(\nabla_h^2 h, (\nabla_h h)^2\right)$ being discarded within the overall approximation process (see Berkhoff 1973; Smith & Sprinks 1975). The 'natural' jump conditions associated with (3.7) are merely $[\phi_0] = [\boldsymbol{n} \cdot \nabla_h \phi_0] = 0$, which follow from the application of the limiting process described above, and solutions of that equation have accordingly been sought which are continuous and have continuous first derivatives. The present approach shows that $[\phi_0] = 0$ and (3.6) are needed to conserve mass flow where $\nabla_h h$ is discontinuous, for both (3.3) and (3.7), and that a smooth approximation to the free surface elevation cannot also be achieved at such places. Numerical results given in §5 compare the effects of using the different jump conditions with (3.7) and confirm the predicted improvement which (3.6) provides.

A related, but simpler, situation arises in the shallow water approximation to plane wave scattering by a vertical step (see, for example, Lamb 1932). Here also, the appropriate matching conditions at the step are continuity of the free surface elevation and of mass flow, and they result in a discontinuous free surface slope. In fact, we can derive the appropriate equations by using the variational approach. If we temporarily suspend the hypothesis that h is continuous and use the trial function $\psi(x, y, z) = \chi(x, y)$, say, we find that the natural conditions of $\delta \mathscr{L} = 0$ are the familiar shallow water equation $\nabla_h \cdot h \nabla_h \chi + v\chi = 0$ in $D_- \cup D_+$, with $[h \boldsymbol{n}_\Gamma \cdot \nabla_h \chi] = 0$ on Γ, which represents continuity of mass flow at places where h is discontinuous. While the local solution at the step is not physically plausible, the scattered wave amplitudes are correct in the long wave limit, as shown by Bartholomeusz (1958). A similar conclusion follows from our numerical solutions of (3.3) and (3.7).

4. An extended approximation

We now consider the effect of using a generalization of (3.1) to approximate ϕ, taking

$$\psi(x, y, z) = \frac{\mathrm{i}g}{\sigma} \sum_{n=0}^{N} w_n(h, z) \phi_n(x, y) . \tag{4.1}$$

The functions w_n are assumed to be assigned but we do not need to specify them for the moment. We shall, however, return to (3.1) to provide the leading term of (4.1).

The functions ϕ_n are to be determined and if we use (4.1) in $L(\psi)$ we obtain a functional $L_2(\phi_0, ..., \phi_N)$, say, whose first variation can be inferred from (2.3) in the form

$$\delta L_2 = \frac{\mathrm{i}g}{\sigma} \sum_{n=0}^{N} \left\{ \iint_D \delta\phi_n \left(-(w_n(\psi_z - v\psi))_{z=0} + (w_n(\psi_z + \nabla_h h \cdot \nabla_h \psi))_{z=-h} \right.\right.$$
$$\left.\left. + \int_{-h}^{0} w_n \nabla^2 \psi \mathrm{d}z \right) \mathrm{d}x\, \mathrm{d}y + \int_C \boldsymbol{n} \cdot \delta\phi_n \int_{-h}^{0} w_n \nabla_h \psi \mathrm{d}z \,\mathrm{d}C \right\},$$

where ψ is given by (4.1). It follows that L_2 is stationary for independent variations

$\delta\phi_n$ all of which vanish on C if and only if

$$\sum_{m=0}^{N}\left\{\int_{-h}^{0} w_n\nabla^2(w_m\phi_m)\,\mathrm{d}z+(\nabla_h h\cdot\nabla_h\phi_m)(w_nw_m)_{z=-h}\right.$$
$$\left.+\Big(\big(w_n((w_m)_z+\nabla_h h\cdot\nabla_h w_m)\big)_{z=-h}-\big(w_n((w_m)_z-\nu w_m)\big)_{z=0}\Big)\phi_m\right\}=0, \qquad (4.2)$$

for $n=0,...,N$.

If we substitute (4.1) into $\mathscr{L}(\psi)$ instead and enforce continuity of each ϕ_n across Γ, we obviously obtain the jump conditions

$$\boldsymbol{n}_\Gamma\cdot\sum_{m=0}^{N}\left[\int_{-h}^{0} w_n\nabla_h(w_m\phi_m)\,\mathrm{d}z\right]=0 \qquad (n=0,...,N) \qquad (4.3)$$

to be applied at locations where the bed slope is discontinuous.

The approach so far is quite general but we impose certain conditions on the functions w_n at this stage, namely that each satisfies

$$(w_n)_z-\nu w_n=0 \quad (z=0)\,, \qquad (w_n)_z=0 \quad (z=-h) \qquad (4.4)$$

for all points (x,y) and that

$$\int_{-h}^{0} w_nw_m\mathrm{d}z=0 \quad (m\neq n)\,, \qquad (4.5)$$

also at each (x,y). After some manipulation we then find that the system of equations (4.2) can be written as

$$\nabla_h\cdot\int_{-h}^{0} w_n{}^2\mathrm{d}z\nabla_h\phi_n+\sum_{m=0}^{N}\left\{\nabla\phi_m\cdot\int_{-h}^{0}(w_n\nabla_h w_m-w_m\nabla_h w_n)\,\mathrm{d}z\right.$$
$$\left.+\phi_m\Big(\int_{-h}^{0} w_n\nabla_h^2 w_m\mathrm{d}z+\big(w_n\nabla_h h\cdot\nabla_h w_m\big)_{z=-h}\Big)\right\}=0,$$

for $n=0,...N$ and that (4.3) becomes

$$\boldsymbol{n}_\Gamma\cdot\left[\nabla_h\phi_n\int_{-h}^{0} w_n{}^2\mathrm{d}z+\sum_{m=0}^{N}\phi_m\int_{-h}^{0} w_n\nabla_h w_m\mathrm{d}z\right]=0 \qquad (n=0,...,N).$$

A more convenient form of these equations is obtained through use of the identity $\nabla_h w_n=(\partial w_n/\partial h)\nabla_h h$, followed by some rearrangement, to give the system of differential equations

$$\nabla_h\cdot a_n\nabla_h\phi_n+\sum_{m=0}^{N}\left\{(b_{mn}-b_{nm})\nabla_h h\cdot\nabla_h\phi_m+\left(b_{mn}\nabla_h{}^2h+c_{mn}(\nabla_h h)^2+d_{mn}\right)\phi_m\right\}=0 \qquad (4.6)$$

for $n=0,...N$ and the system of jump conditions

$$[\boldsymbol{n}_\Gamma\cdot\nabla_h\phi_n]\,a_n+[\boldsymbol{n}_\Gamma\cdot\nabla_h h]\sum_{m=0}^{N} b_{mn}\phi_m=0 \qquad (n=0,...,N). \qquad (4.7)$$

Here we have introduced the functions

$$a_n(h) = \int_{-h}^{0} w_n^2 \mathrm{d}z, \quad b_{mn}(h) = \int_{-h}^{0} \frac{\partial w_m}{\partial h} w_n \mathrm{d}z, \quad c_{mn}(h) = b_{mn}' - \int_{-h}^{0} \frac{\partial w_m}{\partial h} \frac{\partial w_n}{\partial h} \mathrm{d}z, \tag{4.8}$$

with b_{mn}' denoting $\mathrm{d}b_{mn}/\mathrm{d}h$, and

$$d_{mn}(h) = \int_{-h}^{0} w_n \frac{\partial^2 w_m}{\partial z^2} \mathrm{d}z \ .$$

We now make a particular choice of the functions w_n which extends (3.1) in a natural way, namely

$$w_n(h,z) = \sec(k_n h) \cos(k_n(z+h)) \qquad (n = 0, ..., N), \tag{4.9}$$

where $k_1(h), ..., k_N(h)$ denote the positive, real roots of

$$k_n \tan(k_n h) = -\nu \ , \tag{4.10}$$

arranged in ascending order of magnitude, and $k_0(h) = \mathrm{i}k(h)$, which subsumes the function w_0 of (3.1) into the set (4.9). This set satisfies (4.4) and (4.5) and can therefore be used in (4.6) and (4.7). The minor simplification $d_{nn} = -k_n^2 a_n$, $d_{mn} = 0$ $(m \neq n)$ can be made in (4.6) as a result of the choice of (4.9).

For this choice, the system (4.6) is a generalization of the modified mild-slope equation (3.3) and (4.7) is the corresponding extension of the mass flow jump condition (3.6). The correspondence between the notation of the present section and that of §3, which is deliberately aligned with Chamberlain & Porter (1995*a*), is $a_0 = u_0$, $b_{00} = u_1$ and $c_{00} = u_2$. If the terms $O\left(\nabla_h^2 h, (\nabla_h h)^2\right)$ are neglected in (4.6), on the basis of the mild-slope approximation, the simplified system which results may be regarded as an extension of the mild-slope equation of Berkhoff (1973). However, there seems to be little justification in seeking a higher-order approximation and discarding some of the resulting terms.

We now restrict attention to two-dimensional scattering problems, such as that considered by Booij (1983) to examine the mild-slope equation. We therefore let $h = h(x)$ and $\phi_n = \phi_n(x)$ $(n = 0, ..., N)$ so that (4.6) and (4.7) reduce to

$$\left(a_n \phi_n'(x)\right)' - k_n^2 a_n \phi_n(x) + \sum_{m=0}^{N} \Big\{ (b_{mn} - b_{nm}) h'(x) \phi_m'(x) + \left(b_{mn} h''(x) + c_{mn} \left(h'(x)\right)^2\right) \phi_m(x) \Big\} = 0, \tag{4.11}$$

for $n = 0, ...N$, and

$$\left[\phi_n'(x)\right] a_n + \left[h'(x)\right] \sum_{m=0}^{N} b_{mn} \phi_m(x) = 0 \qquad (n = 0, ..., N), \tag{4.12}$$

respectively.

On an interval where h is constant, (4.11) is simply $\phi_n'' - k_n^2(h)\phi_n = 0$ $(n = 0, ..., N)$, so that ϕ_0 is a linear combination of the propagating wave modes $\exp(\pm \mathrm{i}k(h)x)$ and, for $n \geqslant 1$, ϕ_n is a linear combination of the evanescent modes $\exp(\pm k_n(h)x)$. From this point of view, the choice (4.9) of the expansion set is particularly appropriate for the bed profile $h = h(x)$.

For the purpose of numerical calculations, explicit expressions are required for the coefficients defined by (4.8). We find from (4.9) that

$$\frac{\partial w_n}{\partial h}(h,z) = -k_n'(h)\sec(k_n h)\left[z\sin(k_n(z+h)) - k_n^{-1}\sin(k_n h)\sin(k_n z)\right]$$

where, from (4.10),

$$k_n'(h) = -2k_n^2\,(2k_n h + \sin(2k_n h))^{-1} \ .$$

Straightforward, if tedious, calculations then give

$$a_n = \frac{1}{2k_n}\tan(k_n h)\left(1 + \frac{2k_n h}{\sin(2k_n h)}\right),$$

$$b_{mn} = \begin{cases} \sec(k_n h)\sec(k_m h)\left(\dfrac{k_m^2}{k_n^2 - k_m^2}\right) & (m \neq n)\,, \\[2ex] \dfrac{\sec^2(k_n h)}{4\,(K_n + \sin(K_n))}\left[\sin(K_n) - K_n\cos(K_n)\right] & (m = n)\,, \end{cases}$$

$$c_{mn} = \frac{-2k_m\sec(k_n h)\sec(k_m h)}{2k_m h + \sin(2k_m h)}\left(\frac{4k_m^2 k_n^2 + (k_m^4 - k_n^4)\sin^2(k_m h)}{(k_n^2 - k_m^2)^2}\right) \qquad (m \neq n)\,,$$

$$c_{nn} = \frac{-k_n\sec^2(k_n h)}{12\,(K_n + \sin(K_n))^3}\Big[K_n^4 + 4K_n^3\sin(K_n) + 9\sin(K_n)\sin(2K_n)$$
$$-3K_n\big(K_n + 2\sin(K_n)\big)\big(\cos^2(K_n) - 2\cos(K_n) + 3\big)\Big],$$

where $K_n = 2k_n h$. We have already noted that the coefficients occurring in (3.3) and (3.6) are included in this set.

Massel (1993) has used Galerkin's method to implement the particular approximation resulting from the expansion set (4.9). However, he derives the matching conditions incorrectly with the result that he fails to ensure continuity of mass flow at the interface between two domains. Although the variational approach developed in the present paper gives the clearest view of the approximation process and its relationship to physical principles, Staziker (1995) has obtained the correct general matching conditions (4.12) for two-dimensional scattering by using a Galerkin weak form. We note that Massel (1993) does not give numerical results for $N > 0$.

5. Applications and numerical results

As our principal aim is to re-assess Booij's (1983) results in the light of the mass-conserving jump condition and the extended approximation, we consider the scattering of a plane harmonic wave train by the bed profile $h = h(x)$, where

$$h(x) = \begin{cases} h_0, & x \leqslant 0\,, \\ h_1, & x \geqslant l\,, \end{cases}$$

h_0, h_l and l being given constants.

We have already noted the simple form taken by the solutions of (4.11) on an interval where the bed is horizontal and accordingly set

$$\left.\begin{aligned} \phi_0(x) &= e^{ik(h_0)x} + Re^{-ik(h_0)x} \\ \phi_n(x) &= A_n e^{k_n(h_0)x} \qquad (n = 1, ..., N) \end{aligned}\right\} \quad (x < 0) \tag{5.1}$$

and

$$\left.\begin{aligned} \phi_0(x) &= Te^{ik(h_1)(x-l)} \\ \phi_n(x) &= B_n e^{-k_n(h_1)(x-l)} \qquad (n = 1, ..., N) \end{aligned}\right\} \quad (x > l). \tag{5.2}$$

We have restricted attention to a wave incident from the left, for definiteness, but only trivial amendments are required to include a wave incident from the right.

The principle unknowns are the complex amplitudes R and T of the scattered waves. These are obtained by finding that solution of (4.11) for $0 < x < l$ which matches correctly with (5.1) and (5.2). Assuming that $h'(x)$ is discontinuous at both $x = 0$ and $x = l$ (as is the case in Booij's problem) the matching conditions to be applied at these locations are $[\phi_n] = 0$ for $n = 0, ..., N$ and (4.12). Noting from (5.1) that $\phi_0{}'(0-) + ik(h_0)\phi_0(0) = 2ik(h_0)$ and that $\phi_n{}'(0-) - k_n(h_0)\phi_n(0) = 0$ for $n = 1, ...N$, we find that

$$\left.\begin{aligned} a_0\phi_0{}'(0+) + h'(0+)\sum_{m=0}^{N} b_{m0}\phi_m(0) + ia_0k(h_0)\phi_0(0) &= 2ia_0k(h_0), \\ a_n\phi_n{}'(0+) + h'(0+)\sum_{m=0}^{N} b_{mn}\phi_m(0) - a_nk_n(h_0)\phi_n(0) &= 0 \quad (n = 1, ..., N), \end{aligned}\right\} \tag{5.3}$$

where the coefficients a_n and b_{mn} are evaluated at $h = h_0$. Similarly,

$$\left.\begin{aligned} a_0\phi_0{}'(l-) + h'(l-)\sum_{m=0}^{N} b_{m0}\phi_m(l) - ia_0k(h_1)\phi_0(l) &= 0, \\ a_n\phi_n{}'(l-) + h'(l-)\sum_{m=0}^{N} b_{mn}\phi_m(l) + a_nk_n(h_1)\phi_n(l) &= 0 \quad (n = 1, ..., N), \end{aligned}\right\} \tag{5.4}$$

where the coefficients are evaluated at $h = h_1$.

Together, (5.3) and (5.4) provide $2N + 2$ boundary conditions to be associated with (4.11), holding on $0 < x < l$. In the case of the modified mild-slope equation (3.3) (corresponding to $N = 0$) these conditions take the form

$$\left.\begin{aligned} u_0(h_0)\phi_0{}'(0+) + \{h'(0+)u_1(h_0) + iu_0(h_0)k(h_0)\}\,\phi_0(0) &= 2iu_0(h_0)k(h_0)\,, \\ u_0(h_1)\phi_0{}'(l-) + \{h'(l-)u_1(h_1) - iu_0(h_1)k(h_1)\}\,\phi_0(l) &= 0, \end{aligned}\right\} \tag{5.5}$$

when the notation of (3.3) is reinstated. In all cases the required scattered wave amplitudes are recovered by using $R = \phi_0(0) - 1$ and $T = \phi_0(l)$.

Details of the numerical procedure used to complete the solution of the problem may be found in Staziker (1995) and it is sufficient to describe the main features here. By writing $\psi_n = \phi_n{}'$ for $n = 0, ...N$, (4.11) can be replaced by an equivalent first-order system of the form $\boldsymbol{\phi}'(x) = \boldsymbol{A}(x)\boldsymbol{\phi}(x)$ where $\boldsymbol{\phi} = (\phi_0, ..., \phi_N, \psi_0, ..., \psi_N)^T$ and $\boldsymbol{A}$ is a $(2N+2) \times (2N+2)$ matrix. Any solution of this system can be written in terms of the $2N+2$ linearly independent functions defined by the initial value problems

$\boldsymbol{\phi}_j' = \boldsymbol{A}\boldsymbol{\phi}_j$, $\boldsymbol{\phi}_j(0) = (0, ..., 0, 1, 0, ..., 0)$ $(j = 1, ..., 2N+2)$, the only non-zero element in the initial value of $\boldsymbol{\phi}_j$ occurring in the jth component. In particular, the solution of (4.11), (5.3) and (5.4) can be expressed as a linear combination of the vectors $\boldsymbol{\phi}_j$ $(j = 1, ..., 2N+2)$, and the values of $\phi_0(0)$ and $\phi_0(l)$ thereby determined. The initial value problems are solved by using the default numerical solution method for first-order systems of differential equations on MATLAB. This numerical method is a fifth-order Runge–Kutta procedure given by Fehlberg (1970) and details of how it is implemented on MATLAB can be found in Forsythe, Malcolm & Moler (1977). Numerical calculations have shown that this method is sufficiently robust for the parameter values which we consider here.

Booij's problem corresponds to the bed profile

$$h(x) = h_0\left(1 - \frac{2x}{3l}\right) \qquad (0 \leqslant x \leqslant l),$$

so that $h_1 = h_0/3$. Results are given for the (real) reflected wave amplitude $|R|$, plotted against the dimensionless slope length $W_s = \nu l$ with the parameter νh_0 held constant and equal to 0.6. Thus $W_s \to 0$ corresponds to $l/h_0 \to 0$.

We first give two graphs which are obtained by using the matching conditions $[\phi_n] = 0$ and $[\phi_n'] = 0$ at $x = 0$ and $x = l$. Figure 1(a) contains the information first given by Booij (1983), superimposed on which is the graph of $|R|$ following from use of the modified mild-slope equation and given by Chamberlain & Porter (1995a). Figure 1(b) shows the effect of the extended approximation, the curves for $N = 1, 2, 3$ corresponding to Massel's (1993) approach (although Massel did not compute the graphs). For clarity, we only show that part of the graph for $N = 3$ where it is noticeably different from that for $N = 2$, and so on. We note that the curves approximating $|R|$ separate from the values given by the unapproximated equations for $W_s \leqslant 1.2$, corresponding to slopes in excess of 1 in 3. This is the basis of Booij's widely quoted upper limit on the gradient allowed by the mild-slope approximation.

The corresponding graphs obtained when the mass-conserving conditions are used at $x = 0$ and $x = l$ are given in figures 2(a) and 2(b). These conditions are (5.5) for the mild-slope equation and its modified version and (5.3) and (5.4) for the extended approximation.

The revised graph based on the mild-slope equation, but used with the 'unnatural' jump conditions (5.5), show that a good approximation is obtained for values $W_s \geqslant 0.4$, corresponding to gradients less than about 1 in 1. Figures 1(a) and 2(a) substantiate our earlier observations that the deletion of the term involving $[\boldsymbol{n} \cdot \nabla_h h]$ is more significant than the approximation of (3.3) by (3.7). The mild-slope and modified mild-slope versions of $|R|$ part company at the same value of W_s in both figures 1(a) and 2(a), and this is where the additional $O(\epsilon^2)$ terms in (3.3) become significant. However, the $O(\epsilon)$ term in (3.6) is responsible for the major differences between the two sets of graphs, for all values of W_s.

We observe that the modified mild-slope equation, used with the correct boundary conditions, gives a good approximation to the full linear version of $|R|$ for quite steep slopes (the smallest value of W_s shown, 0.05, corresponds to a slope which makes an angle in excess of 80° to the horizontal). The use of the extended trial space together with the boundary conditions (5.3) and (5.4) produces a further improvement, but perhaps not enough to justify the additional computational effort required. For other bed geometries, the higher-order approximation is more significant as the further example below illustrates.

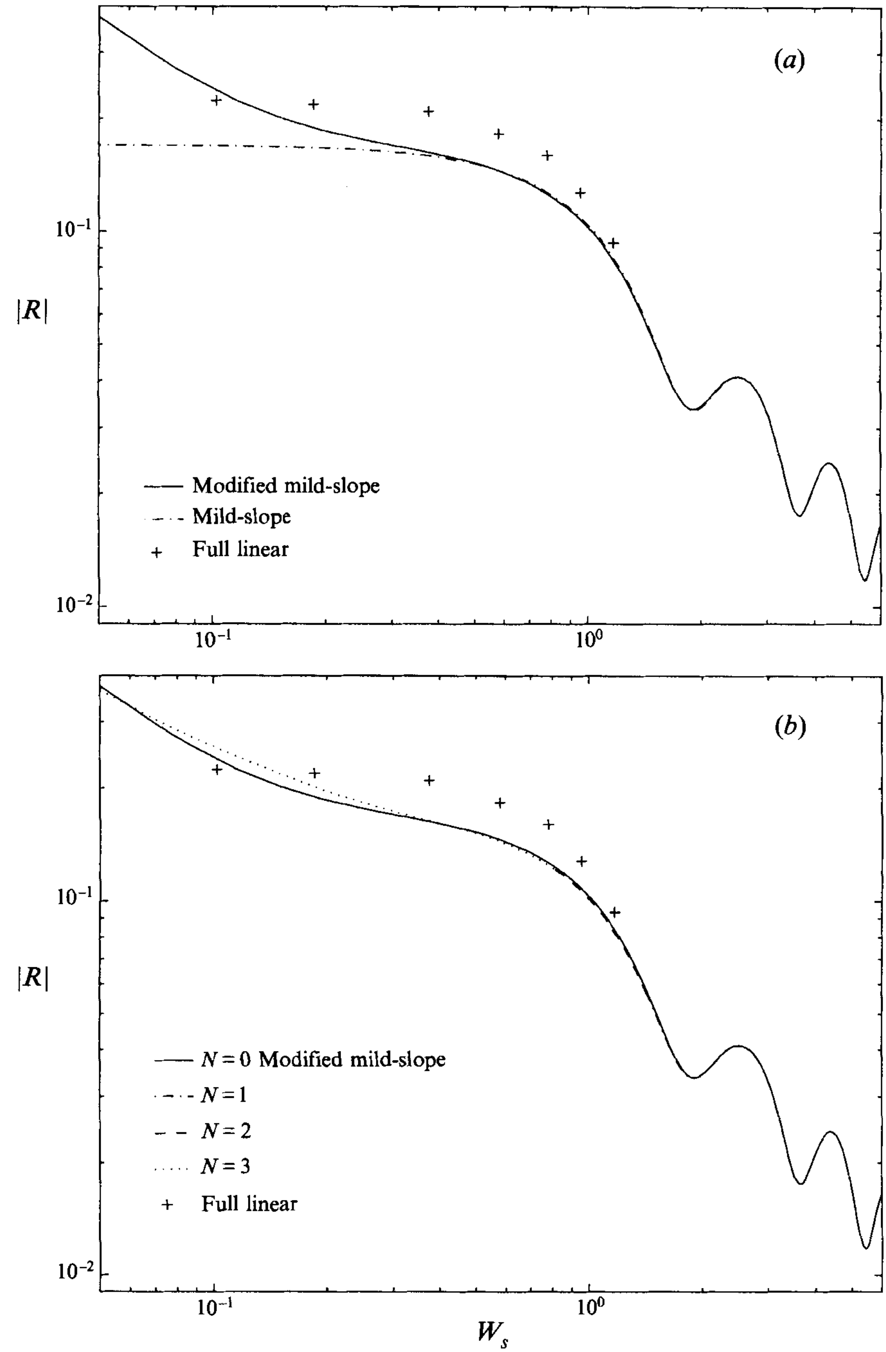


FIGURE 1. (*a*) Comparison of reflected amplitudes for Booij's test problem with a smooth approximation to the free surface. (*b*) As (*a*) but including extended approximations for $N = 1, 2, 3$.

Rey (1992) also considered Booij's problem using a quite different approximation to the full linearized problem. His approach involves approximating the bed profile by a series of horizontal shelves separated by abrupt vertical steps. This series of steps is further subdivided into smaller subsystems of steps called patches. The approximation takes account of evanescent wave modes in a particular way. In each

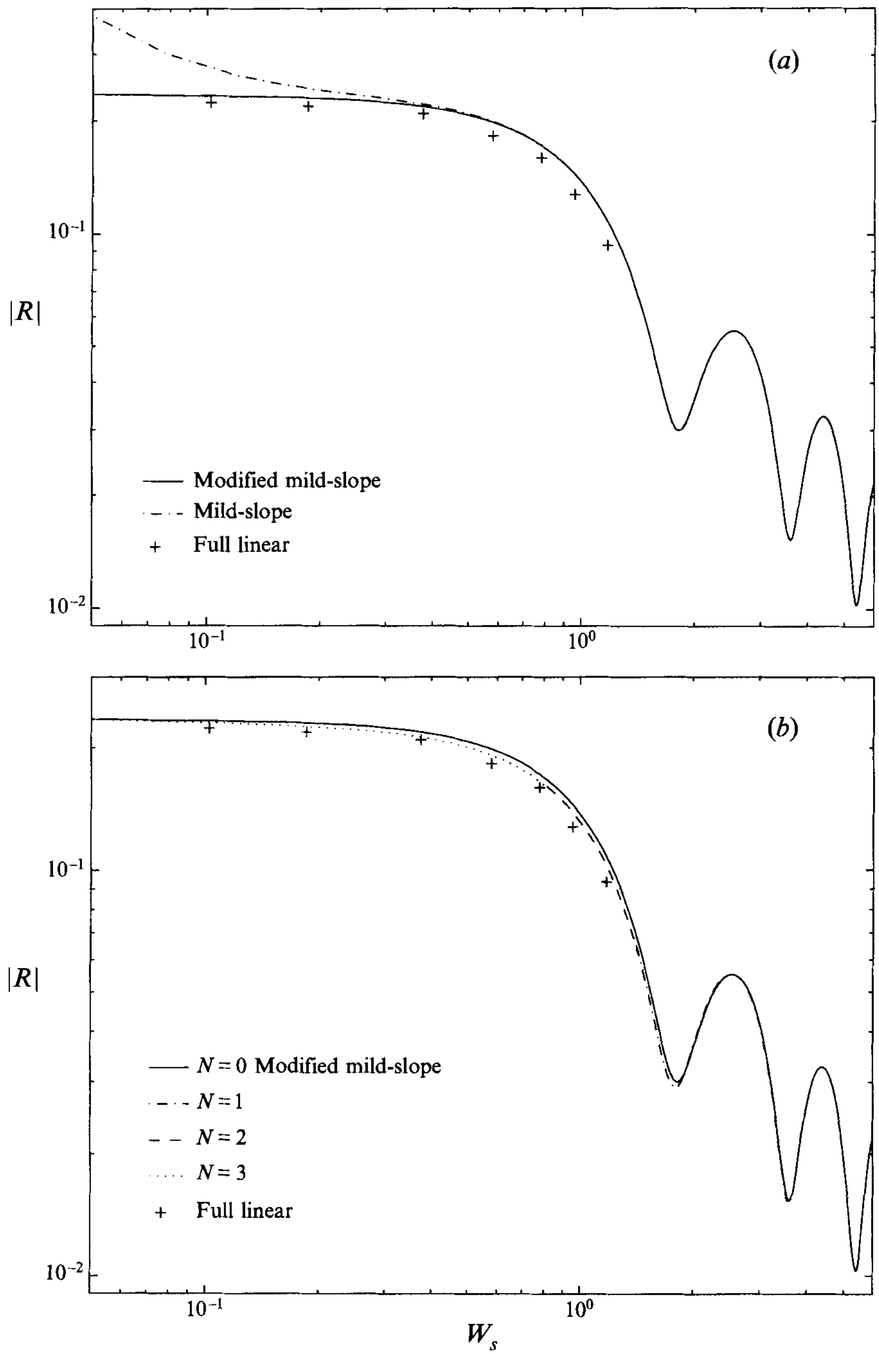
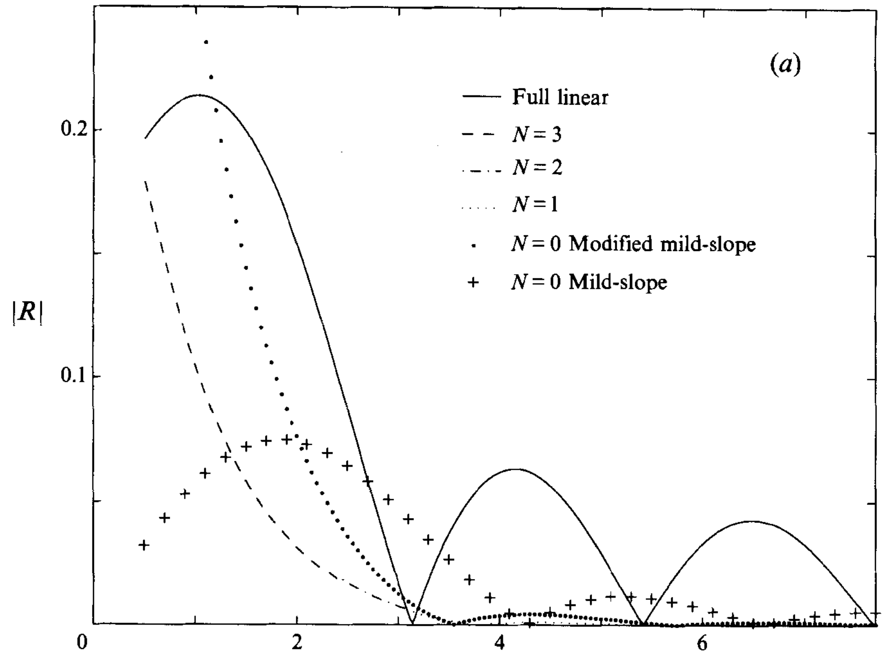


FIGURE 2. (a) Comparison of reflected amplitudes for Booij's test problem with mass conservation imposed. (b) As (a) but including extended approximations for $N = 1, 2, 3$.

patch, the decaying modes generated at one step are not assumed to be negligible at neighbouring steps in the patch. However, the decaying modes generated in one patch are assumed to be negligible at neighbouring patches. The results obtained by Rey using this intricate and computationally expensive method are in good agreement



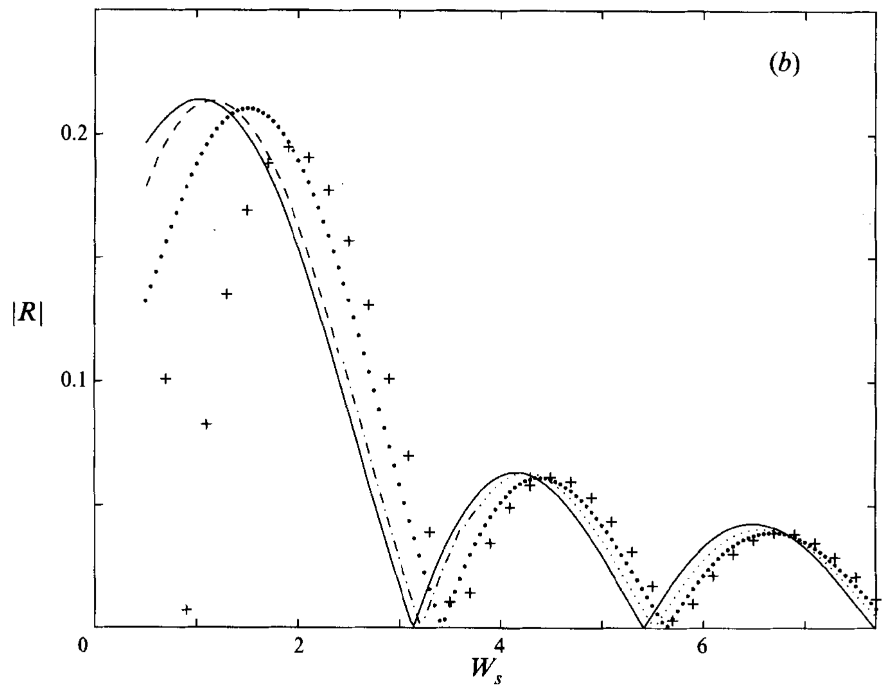


FIGURE 3. Comparison of reflected amplitudes for the elevated bed topography test problem (a) with a smooth approximation to the free surface and (b) with mass conservation imposed.

with those obtained here when the correct mass-conserving boundary conditions at $x = 0$ and $x = l$ are applied.

Figures 3(a) and 3(b) give various approximations of the reflected wave amplitude corresponding to the topography consisting of the symmetric elevation

$$h(x) = h_0 \left\{ 2\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right) + 1 \right\} \qquad (0 \leqslant x \leqslant l),$$

with $h_0 = h_1$. The parameter $W_s = \nu l$, as before, and $\nu h_0 = 1$ is chosen for illustration. In the limit $W_s \to 0$ the topography reduces to a thin vertical barrier of height $\frac{1}{2}h_0$, for which the approximations developed here are inappropriate. In this case the 'exact' (full linear) solution is obtained by an integral equation method which will be reported elsewhere (see Staziker, Porter & Stirling 1995).

The graphs in figure 3(*a*) are obtained by using the matching conditions $[\phi_n] = 0$ and $[\phi_n'] = 0$ at $x = 0$ and $x = l$. Figure 3(*b*) shows the significant effect of replacing the second of these by the mass-conserving jump conditions and that the use of an extended, but small, trial space improves the approximation given by the modified mild-slope equation.

6. Conclusions and remarks

The systematic approach to the mild-slope approximation which variational methods provide, which was used by Chamberlain & Porter (1995*a*) to derive the modified mild-slope approximation, is more fully exploited in the present work. In particular, an interfacial matching condition is derived for the modified mild-slope equation which must be imposed at locations where the bed slope is discontinuous to ensure continuity of mass flow there, at the expense of continuity of free surface slope.

It has been shown that the range of validity of the mild-slope equation is increased by regarding that equation as an approximation to the modified mild-slope equation but retaining the full interfacial condition. This apparently inconsistent coupling has been justified by a quantitative argument, and confirmed by numerical results. Thus, Booij's (1983) estimate of a maximum slope gradient of 1 in 3 is revised to 1 in 1.

In the same paper, Booij (1983) considered the different problem of waves propagating in a direction parallel to the contours of a sloping bed which terminates in vertical walls at its upper and lower edges. For this (eigenvalue) problem where conservation of mass is guaranteed without the need for internal boundaries, Booij concluded that gradients up to 1 in 1 are permissible. We now see that the conflicting maximum gradients resulting from Booij's work are not due to the different types of problem he considered but to the incorrect application of the mild-slope equation to the scattering problem.

In §1 we referred to the extended mild-slope equation devised by Kirby (1986). As pointed out by Chamberlain & Porter (1995*a*), this approximation is something of a hybrid which does not fit neatly into the variational approach, but it is appropriate to mention it again here for the sake of completeness. Kirby defined the bed to be the surface $z = -h(x, y) + \delta(x, y)$, and expanded the bed condition $\phi_z + \nabla_h h \cdot \nabla_h \phi = 0$ to give

$$\phi_z + \nabla_h h \cdot \nabla_h \phi - \nabla_h \cdot \delta \nabla_h \phi = 0 \qquad (z = -h),$$

on the assumption that the term δ represents small-amplitude oscillations about the mean (mild-slope) bed level $z = -h$. Berkhoff's (1973) vertical averaging procedure was then implemented, ϕ being approximated by the function ψ given by (3.1). We infer that the approximate interfacial condition required to ensure continuity of mass flow is the same as that derived for the mild-slope equation, namely (3.6). It should be noted, however, that Kirby's (1986) application of the extended mild-slope equation to ripple beds, in which continuity of the free surface slope is imposed, is correct, for in this case h is regarded as the (constant) mean depth of the ripples and δ alone represents the undulations.

One further remark is in order in relation to ripple beds. We noted earlier that the modified mild-slope equation is successful in predicting the main features of ripple bed scattering, and in particular the resonant peaks, even without the correct matching condition (3.6) (see Chamberlain & Porter 1995*a*). Numerical calculations have shown that, in contrast to the two problems considered above, the matching conditions have virtually no effect in the case of ripple bed problems. This feature can be explained by reference to Chamberlain & Porter (1995*b*), who have analysed the cumulative effect of scattering by ripples and shown that different models of the process are capable of producing very similar results.

A further amplification of the work of Chamberlain & Porter (1995*a*) included here is a higher-order approximation, which extends the original mild-slope concept. The basic variational principle is again invoked to produce a set of coupled partial differential equations, independent of the vertical coordinate, and a consistent mass-conserving interfacial condition. This derivation corrects previous work of Massel (1993) and produces results for scattering by a sloping bed consistent with the computationally more expensive bed discretization method of Rey (1992).

Solutions of the full linear problem for other bed geometries consisting of local elevations are being developed by using integral equation methods (Staziker *et al.* 1995), and these will allow further assessments of the present work to be made.

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